Int. J. Solids Structures, Vol. 17, pp. 379–394, 1981
Printed in Great Britain

0020-7683/81/040379-16$02.00/0
Pergamon Press Ltd.

TORSIONAL WAVES IN AN AXIALLY HOMOGENEOUS BIMETALLIC CYLINDER†

R. K. KAUL and R. P. SHAW
Department of Engineering Science, Aerospace Engineering and Nuclear Engineering, Faculty of Engineering and Applied Sciences, State University of New York at Buffalo, Buffalo, NY 14214, U.S.A.

and

W. MULLER
Department of Mathematics, Christelijke Hogere Technische School, Hilversum, Holland

(*Received 22 April 1980; in revised form 25 August 1980*)

Abstract—The dispersion spectrum is found for axially asymmetric torsional waves in an elastic, bimetallic rod with cylindrical core and concentric outer casing. Plotting of the various branches of the spectrum is simplified by the presence of discrete invariant points which are independent of the material properties and through which the spectral lines must pass. The slope and curvature of the spectral lines at cut-off frequencies, the asymptotic approximations at high frequency, the non-existence of complex branches, the problem of *co-existence* and the concept of energy flow are also studied.

INTRODUCTION

The problem of wave propagation in a piecewise, homogeneous elastic solid is of interest in a number of physical areas, e.g.[1–3]. This study concentrates on a geometrically simple problem, which contains many aspects of more general problems but has not been fully explored in the literature. In particular, the dispersion equation is obtained and analyzed for axially symmetric torsional waves in a bimetallic circular cylinder with solid cylindrical core perfectly bonded to a cylindrical casing of a different elastic material.

Earlier investigations in this area are due to Armenakas[4], Reuter[5] and Haines and Lee[6]. Using Pochhammer's equation for torsional waves in elastic cylinders[7], Armenakas presented numerical results for a few typical choices of material and geometric parameters for real wave numbers only. Reuter carried out an asymptotic analysis for the low frequency long wave length region and determined the expression for the phase velocity of the lowest torsional mode. Haines and Lee reinvestigated the problem and numerically demonstrated the properties of the dispersion spectrum for a general composite cylinder for both real and imaginary wave numbers. They also *formally* pointed out the absence of complex segments of the dispersion spectrum and rederived the low frequency long wave length approximation for the lowest torsional mode.

The present study amplifies and further extends the research carried out by these earlier investigators in this problem area. In addition to a solution for the actual frequency equation, obtained by numerical solution of transcendental eigenvalue equation, a number of other significant results have been obtained. First, it is shown that the eigenvalue equation is separable in the Bernoulli sense so that it may be written as the sum of two analytic functions, one of which is a function of the radial casing wave number and the other is a function of the radial core wave number. The separation constant Q, represents the elastic coupling at the bonded interface between the core and casing and therefore can be incorporated in the interface continuity equations. The solutions to the eigenvalue problem is thus an analytic function of the Lie-parameter Q. In particular, the lower limiting value $Q \to 0^+$ corresponds to a traction-free boundary eigenvalue problem and the upper limiting value $Q \uparrow \infty$ corresponds to a displacement-free boundary eigenvalue problem. The problem of perfect bonding at the interface is therefore an *intermediate* eigenvalue problem. It therefore follows from Weyl–Courant minimax theorem in the theory of self-adjoint differential operators that the Rayleigh quotient is a continuously increasing function of the Lie-parameter Q, and therefore the eigenvalues of the intermediate problem are nested by the eigenvalues of the two limiting cases,

†The support of the Office of Naval Research, Structural Mechanics, contract N00014-75-C-0302, and the National Science Foundation, grant GK40020, for this research is gratefully acknowledged.

that is, $\omega_{Q\to 0^+} < \omega < \omega_{Q\uparrow\infty}$. The dispersion equations for these limiting cases take on a very simple form and can be easily plotted and therefore serve as bounds on the dispersion spectrum of the intermediate problem, as required by the minimax theorem.

To qualitatively sketch the dispersion spectrum, we have also determined analytically the cut-off frequencies, their slope and their curvature at long wave length. We have also shown that except for the lowest torsional mode, coincidence of frequencies at cut-off does not exist and therefore, in general, all branches except the lowest, have zero slope at infinite wave length. The lowest torsional mode is, however, an exception because at cut-off, zero frequency is a non-simple double root of the eigenvalue equation and two linearly independent eigenfunctions can co-exist at this frequency, giving rise to coincidence phenomena, and non-zero slope. We have further shown that in the long wave length region, the curvature of the branch is greater (less) than c_2^2/ω when μ_1/μ_2 is greater (less) than ρ_1/ρ_2. Thus for a typical set of material parameters, all branches of the spectrum except the lowest, have similar geometric properties in the neighborhood of cut-off frequencies. To obviate elaborate computations in the high frequency region we have also obtained McMahon-asymptotic representations of the various frequency equations which enter in the discussion. We finally conclude this study with a brief discussion of group velocity and flow of energy flux in bimetallic cylinder.

BASIC EQUATIONS

For torsional waves in a homogeneous, isotropic, elastic medium the displacement equation of motion in an open domain R is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)u - \frac{1}{r^2}u + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad c^2 > 0 \tag{1}$$

where $c^2 \equiv \mu/\rho$ is the square of wave-speed of the shear waves in an infinite, homogeneous, isotropic elastic medium with rigidity modulus μ and mass density ρ. The tangential displacement component is $u \equiv u_\theta(r, z; t)$, where r is the radial direction, z the longitudinal direction and t is the time parameter, e.g. [7]. Equation (1) is a two-dimensional wave equation and admits plane waves propagating in the longitudinal z-direction. Assuming solutions of the form

$$u(r, z; t) = V(r)\exp i(\xi z - \omega t), \qquad (i \equiv \sqrt{-1}) \tag{2}$$

we find that $V(r)$ must satisfy the equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)V - \frac{1}{r^2}V + \kappa^2 V = 0, \qquad \kappa^2 \gtreqless 0 \tag{3}$$

where

$$\kappa^2 \equiv (\omega/c)^2 - \xi^2, \tag{4}$$

is the square of radial wave number, ξ is the wave number in the longitudinal direction, ω is the angular frequency with real period $2\pi/\omega$ and frequency $\omega/2\pi$. The radial function $V(r)$ is the solution of Bessel equation of order one when $\kappa \neq 0$, and in the limiting case when $\kappa = 0$, the Bessel equation reduces to an Euler equation. In order to accommodate the limiting case when the radial wave number approaches the value zero, we take the general solution of eqn (1) in the form

$$u_\theta = \left[A\frac{2}{\kappa}J_1(\kappa r) - \frac{\pi}{2}\kappa B Y_1(\kappa r)\right]\exp i(\xi z - \omega t), \tag{5}$$

where A and B are suitable coefficients to be determined from the boundary conditions.

We define our bimetallic composite cylinder $R: R_1 \times R_2$ to be the union of two cylinders R_1 and R_2 such that

$$R_1:\ 0 \le r \le r_0, \quad R_2:\ r_0 < r \le a, \quad |z| < \infty$$

$a > r_0$ with interface $r = r_0$, which is assumed to be perfectly bonded. The material properties of the two cylinders are assumed to be

$$R_1\colon\ (\mu_1, \rho_1), \qquad R_2\colon\ (\mu_2, \rho_2).$$

The shear stresses are given by

$$\tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z}, \qquad \tau_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{1}{r} u_\theta\right), \tag{6}$$

and therefore using eqn (5) we find that

$$\tau_{z\theta} = i\mu\xi \left[A \frac{2}{\kappa} J_1(\kappa r) - \frac{\pi}{2} \kappa B Y_1(\kappa r)\right] \exp i(\xi z - \omega t),$$

$$\tau_{r\theta} = -\mu \left[2AJ_2(\kappa r) - \frac{\pi}{2} \kappa^2 B Y_2(\kappa r)\right] \exp i(\xi z - \omega t). \tag{7}$$

For a solid cylinder in region R_1, the boundary conditions as $r \to 0^+$ require that $|u_\theta^{(1)}(0^+, z; t)|$ and $|\tau_{r\theta}^{(1)}(0^+, z; t)|$ be bounded and therefore suitable forms for displacement and shear stress are

$$u_\theta^{(1)} = \frac{2}{\kappa_1} A_1 J_1(\kappa_1 r) \exp i(\xi z - \omega t), \qquad \tau_{r\theta}^{(1)} = -2\mu_1 A_1 J_2(\kappa_1 r) \exp i(\xi z - \omega t), \tag{8}$$

where

$$\kappa_1 \equiv [(\omega/c_1)^2 - \xi^2]^{1/2}, \qquad c_1^2 \equiv \mu_1/\rho_1. \tag{9}$$

For a hollow cylinder in region R_2, the suitable forms for displacement and shear stress are

$$u_\theta^{(2)} = \left[\frac{2}{\kappa_2} A_2 J_1(\kappa_2 r) - \frac{\pi}{2} \kappa_2 B_2 Y_1(\kappa_2 r)\right] \exp i(\xi z - \omega t),$$

$$\tau_{r\theta}^{(2)} = -\mu_2 \left[2A_2 J_2(\kappa_2 r) - \frac{\pi}{2} \kappa_2^2 B_2 Y_2(\kappa_2 r)\right] \exp i(\xi z - \omega t), \tag{10}$$

where

$$\kappa_2 \equiv [(\omega/c_2)^2 - \xi^2]^{1/2}, \qquad c_2^2 \equiv \mu_2/\rho_2. \tag{11}$$

FREQUENCY EQUATION

For a bimetallic cylinder with generators parallel to the z-axis the boundary, interface and boundedness conditions are

$$\begin{aligned}
&1.\ \tau_{r\theta}^{(2)}|_{\partial R_2} = 0 \qquad \partial R_2\colon\ (r = a) \times z,\\
&2.\ \begin{matrix} \tau_{r\theta}^{(1)}|_{\partial R_1} = \tau_{r\theta}^{(2)}|_{\partial R_2} \\ u_\theta^{(1)}|_{\partial R_1} = u_\theta^{(2)}|_{\partial R_2} \end{matrix} \qquad \begin{matrix}\partial R_1 \\ \partial R_2\end{matrix}\colon\ (r = r_0) \times z,\\
&3.\ |u_\theta^{(1)}|_{r\to 0^+} < M, \qquad |\tau_{r\theta}^{(1)}|_{r \to 0^+} < M.
\end{aligned} \tag{12}$$

We assume that R_1 and R_2 are perfectly bonded at their common interface $R_1 \cap R_2$: $(r = r_0) \times z$. We therefore assume that the ensuing motion of the bimetallic cylinder will have a common frequency $\omega/2\pi$ and a common spatial period $2\pi/\xi$ in the axial direction. However, the radial wave numbers $\kappa_1 \in R_1$ and $\kappa_2 \in R_2$ will in general be different. The two wave numbers κ_1 and κ_2 are not independent and only those values are admissible for which the boundary and interface conditions (12) are satisfied. We have one boundary condition $(12)_1$ and two continuity conditions $(12)_2$. From eqns (8), (10) and (12) we find that there are five unknowns A_1, A_2, B_2; κ_1 and κ_2. Since we have only three homogeneous equations relating the five unknowns, it follows that the two radial wave numbers κ_1 and κ_2 are not independent. For the three homogeneous equations to have a non-trivial solution, the determinant of the coefficients A_1, A_2 and B_2 must be zero. This leads to the frequency equation

$$\eta\alpha^2 P_2 - \sigma\gamma[J_2(\gamma)/J_1(\gamma)](\alpha Q_0 + 2S_0) = 0, \tag{13}$$

where

$$\alpha \equiv a\kappa_2, \qquad \gamma \equiv r_0\kappa_1, \qquad \eta \equiv r_0/a, \qquad \sigma \equiv \mu_1/\mu_2, \tag{14}$$

and

$$P_2 \equiv P_0 + \frac{2}{\eta\alpha} Q_0 + \frac{2}{\alpha} R_0 + \frac{4}{\eta\alpha^2} S_0.$$

The functions P_0, P_2; Q_0, R_0 and S_0 are defined as

$$\begin{aligned} P_n &\equiv J_n(\alpha)Y_n(\eta\alpha) - Y_n(\alpha)J_n(\eta\alpha), \qquad n = 0, 2 \\ Q_0 &\equiv J_0(\alpha)Y_0'(\eta\alpha) - J_0'(\eta\alpha)Y_0(\alpha), \\ R_0 &\equiv J_0'(\alpha)Y_0(\eta\alpha) - J_0(\eta\alpha)Y_0'(\alpha), \\ S_0 &\equiv J_0'(\alpha)Y_0'(\eta\alpha) - J_0'(\eta\alpha)Y_0'(\alpha) = P_1, \end{aligned} \tag{15}$$

where primes indicate differentiation with respect to their respective arguments, e.g. [8].

When α is imaginary and γ real, i.e. when $\alpha \to i\alpha$ and $\gamma \to \gamma$, the frequency equation (13) takes the form

$$\eta\alpha^2\tilde{P}_2(\alpha) - \sigma\gamma[J_2(\gamma)/J_1(\gamma)](\alpha\tilde{Q}_0(\alpha) + 2\tilde{S}_0(\alpha)) = 0, \qquad c_2 > c_1 \tag{16}$$

where now α and γ are both real and

$$\begin{aligned} \tilde{P}_2(\alpha) &\equiv I_2(\alpha)K_2(\eta\alpha) - I_2(\eta\alpha)K_2(\alpha), \\ \tilde{Q}_0(\alpha) &\equiv I_0(\alpha)K_1(\eta\alpha) + I_1(\eta\alpha)K_0(\alpha), \\ &\equiv I_0'(\eta\alpha)K_0(\alpha) - I_0(\alpha)K_0'(\eta\alpha), \\ \tilde{S}_0(\alpha) &\equiv I_1(\eta\alpha)K_1(\alpha) - I_1(\alpha)K_1(\eta\alpha), \\ &\equiv I_0'(\alpha)K_0'(\eta\alpha) - I_0'(\eta\alpha)K_0'(\alpha). \end{aligned} \tag{17}$$

When both α and γ are imaginary, i.e. $\alpha \to i\alpha$, $\gamma \to i\gamma$ the frequency equation (13) takes the form

$$\eta\alpha^2\tilde{P}_2(\alpha) + \sigma\gamma[I_2(\gamma)/I_1(\gamma)](\alpha\tilde{Q}_0(\alpha) + 2\tilde{S}_0(\alpha)) = 0, \tag{18}$$

where α and γ are both real in eqn (18), I_n and K_n are modified Bessel functions and other symbols are same as defined earlier.

The case of grazing incidence occurs when either $\gamma = 0$ or when $\alpha = 0$. When $\alpha \to 0$, the frequency equation takes the form

$$\gamma J_2(\gamma) = 0. \tag{19}$$

When $\gamma \to 0$, the frequency equation takes the form

$$\alpha\tilde{P}_2(\alpha) = 0. \tag{20}$$

Equation (20) has no real zeros except $\alpha = 0$, since

$$\tilde{P}_2(\alpha) \equiv I_2(\alpha)K_2(\eta\alpha) - I_2(\eta\alpha)K_2(\alpha),$$
$$\sim 2\cosh\alpha(1-\eta) + \ldots, \qquad \alpha \gg 1, \tag{21}$$

which has no real zeros.† Hence the dispersion spectrum never intersects the line $\gamma = 0$, but approaches it asymptotically from above. This eliminates the existence of imaginary values of α and γ in eqn (13). However, eqn (19) admits *simple* zeros and therefore in the region $\omega/c_2 < \xi < \omega/c_1$, γ is real and α is imaginary.

The last case is when α is real and γ is imaginary, i.e. when $\alpha \to \alpha$ and $\gamma \to i\gamma$. The frequency equation in this case takes the form

$$\eta\alpha^2 P_2(\alpha) - \sigma\gamma[I_2(\gamma)/I_1(\gamma)](\alpha Q_0(\alpha) + 2S_0(\alpha)) = 0, \qquad c_1 > c_2 \tag{22}$$

where now α and γ are both real, and other symbols are as defined earlier. This equation replaces eqn (16) when the wave speed in the core is greater than that in the casing. Such a situation is certainly possible, but is of less physical interest.

ANALYSIS OF FREQUENCY EQUATION

The frequency equation (13) can be treated as a function of two independent variables α and γ and has an interesting property that the equation is separable. It is easy to see that this equation can be written in the form

$$\mathcal{F}(\alpha) = \gamma J_2(\gamma)/J_1(\gamma), \tag{23}$$

where

$$\mathcal{F}(\alpha) \equiv \frac{\eta\alpha^2 P_2(\alpha)}{\sigma[\alpha Q_0(\alpha) + 2S_0(\alpha)]}.$$

Since

$$\frac{d}{d\alpha}\mathcal{F}(\alpha) = 0, \qquad \frac{d}{d\gamma}(\gamma J_2(\gamma)/J_1(\gamma)) = 0, \tag{24‡}$$

it follows that there exists a separation constant Q such that

$$\mathcal{F}(\alpha) - Q = 0, \qquad \gamma J_2(\gamma) - QJ_1(\gamma) = 0, \tag{25}$$

where $0 \le Q < \infty$ and can be considered as a continuous parameter, e.g.[9, 10].

The second of eqn (25) corresponds to the case of a solid cylinder with elastic boundary condition at $r = r_0$

$$\tau_{r\theta}^{(1)}/u_\theta^{(1)} = -Q\mu_1/r_0. \tag{26}$$

When $Q \to 0^+$, we have the case of a traction-free solid cylinder and $Q \uparrow \infty$ corresponds to the case of a solid cylinder with rigid encased surface. The rigid boundary conditions slowly change to traction-free boundary conditions as we gradually change the parameter Q from ∞ to 0^+.

Similarly the first of eqn (25) corresponds to the case of an hollow cylinder with elastic boundary conditions

$$\text{(i)}\ \tau_{r\theta}^{(2)}/u_\theta^{(2)} = -Q\mu_1/r_0, \qquad (r = r_0)\times z$$
$$\text{(ii)}\ \tau_{r\theta}^{(2)} = 0, \qquad (r = a)\times z. \tag{27}$$

†A general proof follows from the theory of Sturm–Liouville equations.
‡Note that this process of separation is capable of generalization.

By varying Q from 0^+ to ∞, the interface boundary condition at $r = r_0$ changes from traction-free to rigid condition. Thus when $Q = 0$, we get from eqn $(25)_1$

$$P_2(\alpha) = 0 \qquad \text{if } \alpha \neq 0, \tag{28}$$

which is the frequency equation of a traction-free hollow cylinder. When $Q \uparrow \infty$, eqn $(25)_1$ gives us the frequency equation

$$\alpha Q_0(\alpha) + 2S_0(\alpha) = 0 \qquad \text{if } \sigma \neq 0, \tag{29}$$

which is the frequency equation of a hollow cylinder with traction-free external surface and rigid internal surface.

We therefore conclude that when $Q \to 0^+$, the two frequency equations are

$$\begin{aligned} &\text{(i) } J_2(\gamma) = 0, \qquad \gamma \neq 0, \\ &\text{(ii) } P_2(\alpha) = 0, \qquad \alpha \neq 0, \end{aligned} \tag{30}$$

and when $Q \uparrow \infty$, the two frequency equations are

$$\begin{aligned} &\text{(i) } J_1(\gamma) = 0, \\ &\text{(ii) } \alpha Q_0(\alpha) + 2S_0(\alpha) = 0 \qquad \text{if } \sigma \neq 0. \end{aligned} \tag{31}$$

It can be easily verified that these frequency equations have only *simple* zeros. Furthermore, these are also the zeros of the general frequency equation (13). In addition these simple zeros have the interesting property that they do not depend upon the choice of the rigidity ratio σ. Thus all of these roots of the frequency equation are invariant with respect to the ratio of the rigidity modulii of the bimetallic cylinder.

Let γ_p, $p = 1, 2, 3, \ldots$ and α_q, $q = 1, 2, 3, \ldots$ be the roots of the two frequency equations (30). Then using these values of α and γ, the spectral branches of the two frequency equations are given by

$$(a\omega/c_2)^2_{pq} = \alpha_q^{\,2} + (a\xi)^2, \qquad (a\omega/c_2)^2_{pq} = \frac{\sigma}{\rho}\,[(\gamma_p/\eta)^2 + (a\xi)^2], \tag{32}$$

where $\rho \equiv \rho_1/\rho_2$. The points of intersection of these branches are

$$(a\xi)^2_{pq} = \frac{\sigma(\gamma_p/\eta)^2 - \rho\alpha_q^{\,2}}{\rho - \sigma}, \qquad \left(\frac{a\omega}{c_2}\right)^2_{pq} = \frac{\sigma[(\gamma_p/\eta)^2 - \alpha_q^{\,2}]}{\rho - \sigma}. \tag{33}$$

From the earlier discussion it is obvious that these points of intersection are invariant points of the dispersion spectrum through which the branches of the frequency equation (13) must pass. Furthermore, these are the only points at which these branches can pass through these $Q \to 0^+$ and $Q \uparrow \infty$ spectral branches which may then be thought of as bounding curves.

For large values of the argument, frequency equations (30) have the Hankel-type asymptotic form

$$\begin{aligned} &\text{(i) } \tan(\gamma - \pi/4) = \frac{8}{15}\,\gamma, \\ &\text{(ii) } \tan\alpha(1-\eta) = \frac{15}{8}\frac{(1-\eta)}{\eta\alpha}. \end{aligned} \tag{34}$$

It is relatively easy to find the roots of these transcendental equations. These roots when used in eqn (33) determine the invariant intersection points of the frequency equation (13) in the high frequency region.

Now consider the remaining two frequency equations (31). For large values of the argument, the Hankel-asymptotic forms for these equations are

$$\begin{aligned}&\text{(i)}\ \tan(\gamma-\pi/4)+\frac{3}{8\gamma}=0,\\&\text{(ii)}\ \tan\alpha(1-\eta)+\frac{8\eta\alpha}{3(1-5\eta)}=0.\end{aligned}\tag{35}$$

Let γ_n, $n=1,2,3,\ldots$ and α_m, $m=1,2,3,\ldots$ be the zeros of these two frequency equations which for large values of γ_n and α_m agree with the zeros of eqn (31). Then in the high frequency region the spectral branches of the two frequency equations (31) are given by eqn (32) if we replace γ_p by γ_n and α_q by α_n. Under the same transformation, the points of intersection of the branches are given by eqn (33). Again, these additional intersection points are invariant points of the spectrum through which the branches of the frequency equation (13) must pass.

Now Q is a continuous parameter and for various values of Q one may easily find the roots of the frequency equation (25). Because the frequency equations are *entire* functions, it is possible to express α and γ in terms of power series in Q and one can then study the properties of the α and γ trajectories with respect to Q. However, such an analysis is not of much interest because of roots α of eqn $(25)_1$ depend upon the choice of ratio σ and the intersection points of the α- and γ-branches are not invariant points of the spectrum. With a view towards usefulness in the study of Floquet waves in bimetallic cylinder with periodic structure, only these invariant points merit special consideration. As such, no attempt is made to study the properties in the neighborhood of the σ-variant intersection points for other values of parameter Q.

LONG WAVELENGTH PROPERTIES

With the invariant points at our disposal, the branches of the spectrum pertaining to the frequency equation (13) can be qualitatively traced if one has: (a) the cut-off frequencies; (b) the slope of the branches at cut-off; and (c) the curvature of the branches at cut-off. Supplementing this information with the properties at long wavelength and low frequency; zeros in the zero-frequency plane; and the zeros at grazing incidence, provides sufficient information to trace the branches of the spectrum qualitatively. Additional information based on the theory of differential equations reveals that the problem of *coexistence* does not exist because all the roots are simple and furthermore, complex roots are inadmissible. However, the lowest torsional mode is an exception.

(i) *Slope at cut-off*

If we represent symbolically the frequency equation (13) by the functional form $F(\alpha,\gamma)=0$, then using the differentiation formulas of implicit functions, we get

$$\frac{d\omega}{d\xi}=-\frac{F_\alpha\alpha_\xi+F_\gamma\gamma_\xi}{F_\alpha\alpha_\omega+F_\gamma\gamma_\omega}\tag{36}$$

where $\alpha\equiv\alpha(\omega(\xi),\xi)$, $\gamma\equiv\gamma(\omega(\xi),\xi)$ and $F_\alpha, F_\gamma, \alpha_\xi,\ldots,\gamma_\omega$ indicate partial derivatives with respect to the variables $\alpha,\gamma,\xi,\ldots$, and ω, respectively. Now

$$\alpha_\xi=-a^2\xi/\alpha,\qquad \gamma_\xi=-r_0{}^2\xi/\gamma,\tag{37}$$

and in the limit as $\xi\to0$, $\alpha_\xi|_0\to0$ and $\gamma_\xi|_0\to0$. Therefore it follows that at cut-off frequencies, the slope of the branches is in general zero, i.e.

$$\left.\frac{d\omega}{d\xi}\right|_{\xi\to0}=0,\tag{38}$$

except when the denominator $(F_\alpha\alpha_\omega+F_\gamma\gamma_\omega)=0$. For non-zero values of wave number ξ the slope of the branches will vanish if

$$\frac{1}{\alpha}\frac{\partial F}{\partial \alpha}+\eta^2\frac{1}{\gamma}\frac{\partial F}{\partial \gamma}=0. \tag{39}$$

In the region of long wavelength and low frequency, when both ξ and ω tend to zero, eqn (36) takes an indeterminate form, since it can be rewritten as

$$\frac{d\omega}{d\xi}=\left(\frac{\gamma F_\alpha+\eta^2\alpha F_\gamma}{\gamma c_1{}^2F_\alpha+\eta^2c_2{}^2\alpha F_\gamma}\right)\frac{\xi}{\omega}. \tag{40}$$

This can be written as

$$\frac{d(\omega)^2}{d(\xi)^2}=\frac{\gamma F_\alpha+\eta^2\alpha F_\gamma}{\gamma c_1{}^2F_\alpha+\eta^2c_2{}^2\alpha F_\gamma},$$

which suggests that in the low frequency long wave length region, ω^2 is linearly proportional to ξ^2. To determine the slope of the branch in this region, when both $\omega \ll 1$ and $\xi \ll 1$, we use power series expansion of Bessel functions for fixed order and small values of the argument. The frequency equation in this region takes the approximate form

$$\sigma\eta\gamma^2+(\eta^{-1}-\eta^3)\alpha^2\approx 0,$$

or equivalently

$$\left(\frac{\omega}{\xi}\right)^2=c_2{}^2\left(1+\frac{(\sigma-\rho)\eta^4}{1-(1-\rho)\eta^4}\right), \tag{41}$$

as conjectured earlier. From here we immediately conclude that the slope of the lowest branch at long wave length and low frequency is given by

$$\left.\frac{d\omega}{d\xi}\right|_{R_0}=c_2\left(1+\frac{(\sigma-\rho)\eta^4}{1-(1-\rho)\eta^4}\right)^{1/2}, \qquad R_0\colon \begin{matrix}\omega\ll 1\\ \xi\ll 1\end{matrix} \tag{42}$$

which agrees with the limiting value obtained by Reuter[5].

From eqn (36) we also see that the slope is indeterminate when

$$F_\alpha\alpha_\xi+F_\gamma\gamma_\xi=F_\xi=0, \qquad F_\alpha\alpha_\omega+F_\gamma\gamma_\omega=F_\omega=0.$$

Simultaneous solution of the three equations

$$F=0, \qquad F_\xi=0 \ \text{ and } \ F_\omega=0, \tag{43}$$

determine ω, ξ and σ for which we have coalescence of spectral lines. These are points of coincidence and as shown in Ref. [11] the slope at these points is given by

$$F_{\omega\omega}\left(\frac{d\omega}{d\xi}\right)^2+2F_{\omega\xi}\frac{d\omega}{d\xi}+F_{\xi\xi}=0, \tag{44}$$

which gives us the value of slopes

$$\left(\frac{d\omega}{d\xi}\right)F_{\omega\omega}=\{-F_{\omega\xi}\pm(F^2_{\omega\xi}-F_{\omega\omega}F_{\xi\xi})^{1/2}\}. \tag{45}$$

When $F_{\omega\xi}|_{\xi\to 0}\to 0$, as in the present case, the slope of the branches at points of coincidence takes the simple form

$$\left.\frac{d\omega}{d\xi}\right|_{\xi\to 0} = \sqrt{\left(-\frac{F_{\xi\xi}}{F_{\omega\omega}}\right)}. \tag{46}$$

In the present problem it can be verified that eqn (43) has only one non-trivial solution $(\omega, \xi):(0, 0)$, which reveals that the origin in the (ω, ξ)-plane is a zero of multiplicity 2. In the δ-neighborhood of the origin $F_{\omega\omega} = 2$, $F_{\xi\xi} = -c_2^2[1 + (\sigma - \rho)\eta^4/[1 - (1 - \rho)\eta^4]]$ and therefore from eqn (46) we get $(d\omega/d\xi)_{(0,0)}$, which agrees with the result derived earlier for the lowest branch. As remarked earlier, this multiplicity leads to co-existence of two linearly independent eigenfunctions.

(ii) *Curvature at cut-off*

At cut-off frequencies when the slope is zero, the curvature of the branches is given by

$$\left.\frac{d^2\omega}{d\xi^2}\right|_{\substack{\xi\to 0\\ \omega_\xi\to 0}} = -\left.\left(\frac{F_\alpha\alpha_{\xi\xi} + F_\gamma\gamma_{\xi\xi}}{F_\alpha\alpha_\omega + F_\gamma\gamma_\omega}\right)\right|_{\substack{\xi\to 0\\ \omega_\xi\to 0}}. \tag{47}$$

Now

$$\alpha_{\xi\xi}|_{\xi\to 0} = -ac_2/\omega, \qquad \gamma_{\xi\xi}|_{\xi\to 0} = -r_0c_1/\omega, \qquad \alpha_\omega|_{\xi\to 0} = a/c_2, \qquad \gamma_\omega|_{\xi\to 0} = r_0/c_1,$$

and therefore the curvature of all those branches at cut-off, whose slope is zero, is given by

$$\left.\frac{d^2\omega}{d\xi^2}\right|_{\substack{\xi\to 0\\ \omega_\xi\to 0}} = \frac{c_2^2}{\omega}\left(1 + \frac{\eta(c_1/c_2 - c_2/c_1)}{F_\alpha/F_\gamma + \eta(c_2/c_1)}\right). \tag{48}$$

For high frequencies and long wavelengths, both α and γ are large. Using asymptotic expansion of Bèssel functions of large argument and fixed order, the frequency equation (13) takes the Hankel-asymptotic form

$$\eta\alpha \sin\alpha(1 - \eta)(\cos\gamma - \sin\gamma) + \sigma\gamma\cos\alpha(1 - \eta)(\cos\gamma + \sin\gamma) \approx 0. \tag{49}$$

From here one can easily show that

$$\left(\frac{F_\alpha}{F_\gamma}\right)_{\xi\to 0} \sim \frac{(1-\eta)\{(\sigma^2\gamma^2 + \eta^2\alpha^2) + (\sigma^2\gamma^2 - \eta^2\alpha^2)\sin 2\gamma\} - \eta\sigma\gamma\cos 2\gamma}{\eta\sigma\alpha(2\gamma + \cos 2\gamma)},$$

and

$$\left(\frac{F_\alpha}{F_\gamma} + \eta\frac{c_2}{c_1}\right)_{\xi\to 0} \sim \frac{\eta}{\sigma}\left\{\frac{(1-\eta)\frac{a\omega}{c_2}\{(1+\sigma\rho) + (1-\sigma\rho)\sin 2\gamma_0\} + 2\gamma_0\sqrt{(\sigma\rho)}}{(2\gamma_0 + \cos 2\gamma_0)}\right\}.$$

Therefore the curvature of the higher branches at cut-off is given by

$$\left.\frac{d^2\omega}{d\xi^2}\right|_{\substack{\xi=0\\ \omega_\xi=0\\ \omega\gg 1}} \sim \frac{c_2^2}{\omega}\left(1 + \frac{(\sigma/\rho)^{1/2}(\sigma - \rho)(2\gamma_0 + \cos 2\gamma_0)}{2\gamma_0(\sigma\rho)^{1/2} + (1-\eta)\left(\frac{a\omega}{c_2}\right)\{(1+\sigma\rho) + (1-\sigma\rho)\sin 2\gamma_0}\right), \tag{50}$$

where $\gamma_0 \equiv \eta(\rho/\sigma)^{1/2}(a\omega/c_2)$ and $a\omega/c_2$ is a dimensionless frequency.

Now

$$(1 + \sigma\rho) + (1 - \sigma\rho)\sin 2\gamma_0 = (\cos\gamma_0 + \sin\gamma_0)^2 + \sigma\rho(\cos\gamma_0 - \sin\gamma_0)^2,$$

and is therefore a positive-definite quantity. Also, for $\gamma_0 > (1/2)$ and real, $2\gamma_0 + \cos 2\gamma_0 = (2\gamma_0+1)\cos^2\gamma_0 + (2\gamma_0 - 1)\sin^2\gamma_0 > 0$. Hence the second term in the parentheses is positive if $\sigma > \rho$ and negative if $\sigma < \rho$. When $\sigma = \rho$, the curvature is $c_2{}^2/\omega$, which is the curvature of the α-branch at cut-off frequency. Thus starting at cut-off frequency with zero slope, the branch of the spectrum lies above (below) the α-hyperbola, when σ is greater (less) than ρ.

(iii) *Cut-off frequencies*

To sketch the higher branches qualitatively, it now remains to determine the cut-off frequencies. At $\xi = 0$, these frequencies are given by the higher roots of the frequency equation

$$\tan(1-\eta)\Omega - (\sigma\rho)^{1/2}\tan\left[\eta(\rho/\sigma)^{1/2}\Omega - \frac{\pi}{4}\right] = 0, \tag{51}$$

where $\Omega \equiv a\omega/c_2 = \alpha|_{\xi\to 0} = \dfrac{1}{\eta}(\sigma/\rho)^{1/2}\gamma|_{\xi\to 0}$.

ASYMPTOTIC FORMULA FOR THE ZEROS OF FREQUENCY EQUATION WHEN $|Q|\to\infty$

The zeros of the frequency equation (30) when $Q = 0$ are well tabulated and their asymptotic form can also be found in [8]. Also, one can find the zeros of $J_1(\gamma) = 0$, well tabulated. However asymptotic formulas for the zeros of frequency equation $(31)_2$ are not available and in this section we obtain their McMahon-asymptotic representation [12].

Consider the equation

$$\alpha Q_0(\alpha) + 2S_0(\alpha) = 0, \tag{52}$$

where

$$Q_0(\alpha) \equiv J_1(\eta\alpha)Y_0(\alpha) - J_0(\alpha)Y_1(\eta\alpha), \qquad S_0(\alpha) \equiv Y_1(\eta\alpha)J_1(\alpha) - Y_1(\alpha)J_1(\eta\alpha),$$

and $\eta \equiv r_0/a < 1$. Using Hankel-asymptotic expansion of cylinder functions for fixed order and large arguments, it can be shown that eqn (52) can be written as

$$\cos[(1-\eta)\alpha + \theta] = 0, \tag{53}$$

where

$$\begin{aligned} \cos\theta &\sim 1 + \frac{15}{2(8\eta\alpha)^2}(1 + 6\eta - 7\eta^2) + \dots, \\ \sin\theta &\sim \frac{3}{8\eta\alpha}\left[(5\eta - 1) + \frac{5}{2(8\eta\alpha)^2}(7 + 15\eta + 21\eta^2 + 21\eta^3) + \dots\right]. \end{aligned} \tag{54}$$

The zeros of eqn (53) are given by

$$(1-\eta)\alpha_n = \beta_n - \theta, \tag{55}$$

where

$$\beta_n = (2n-1)\frac{\pi}{2}, \qquad n = 1, 2, 3, \dots \tag{56}$$

and θ is the solution of

$$\tan\theta \sim \frac{3}{8\eta\alpha}\left[(5\eta - 1) + \frac{5}{(8\eta\alpha)^2}(5 + 9\eta - 45\eta^2 + 63\eta^3) + \dots\right]. \tag{57}$$

When $\alpha \gg 1$, it follows that $\tan\theta \ll 1$ and therefore we can write

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \tag{58}$$

From eqns (57) and (58) we therefore get

$$\theta = \frac{3}{8\eta\alpha}(5\eta - 1) + \frac{12}{(8\eta\alpha)^3}(7 - 15\eta^3) + \cdots \tag{59}$$

From eqns (55) and (59) it now follows that for large values of α, the asymptotic value of the roots is governed by the algebraic equation

$$\alpha = \gamma + p/\alpha + q/\alpha^3 + \cdots, \tag{60}$$

where

$$\gamma = \frac{\beta}{1-\eta}, \qquad p = \frac{3(1-5\eta)}{8\eta(1-\eta)}, \qquad q = \frac{12(15\eta^3 - 7)}{(8\eta)^3(1-\eta)}. \tag{61}$$

Using Lagrange's expansion theorem, the roots of the algebraic equation (60) are

$$\alpha \sim \gamma + p/\gamma + (q - p^2)/\gamma^3 + \cdots \tag{62}$$

Hence for large values of α, the asymptotic formula for the zeros of the frequency equation (52) have the explicit representation

$$\alpha_n \sim \gamma_n + \frac{3(1-5\eta)}{8\eta\gamma_n(1-\eta)} - \frac{12(7 - \eta - 60\eta^2 + 135\eta^3 + 15\eta^4)}{(8\eta\gamma_n)^3(1-\eta)^2} + \cdots O(1/\gamma_n^5), \tag{63}$$

where

$$\gamma_n = \frac{(2n-1)}{(1-\eta)}\frac{\pi}{2}. \tag{64}$$

INADMISSIBILITY OF COMPLEX WAVE NUMBER ξ

In this section we show that in the case of a bimetallic rod, the wave number ξ can be real or imaginary, but cannot be complex. This easily follows by using the theory of *singular* Sturm–Liouville eigenvalue problems [13].

Consider eqn (3) which can be written in the self-adjoint form

$$Lu + \kappa^2 ru = 0, \qquad r \in R:\ R_1 \times R_2, \quad \kappa^2 \gtreqless 0 \tag{65}$$

where

$$L \equiv \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{1}{r},$$

and $r = 0$ is a singular point. The domain of the operator L is the class of $C^2(I)$ functions which satisfy the boundary and continuity conditions (12).

Let $u_1(r)$ and $u_2(r)$ be two complex-valued functions which satisfy the equations

$$Lu_1 + \kappa_1^2 ru_1 = 0, \quad R_1:\ 0 < r < r_0; \qquad Lu_2 + \kappa_2^2 ru_2 = 0, \quad R_2:\ r_0 < r < a, \tag{66}$$

where κ_1 and κ_2 are assumed to be complex wave numbers. The associated eigenfunctions u_1

and u_2 are assumed to be complex and therefore we choose

$$u_1 = v_1 + iw_1, \quad u_2 = v_2 + iw_2, \quad \kappa_1 = \eta_1 + i\nu_1, \quad \kappa_2 = \eta_2 + i\nu_2,$$
$$\kappa_1^2 = (\eta_1^2 - \nu_1^2) + 2i\eta_1\nu_1, \quad \kappa_2^2 = (\eta_2^2 - \nu_2^2) + 2i\eta_2\nu_2. \tag{67}$$

Substituting into eqn (66) and equating real and imaginary parts, we get the four equations

$$\left.\begin{aligned} Lv_1 + (\eta_1^2 - \nu_1^2)rv_1 - 2\eta_1\nu_1 rw_1 = 0, \\ Lw_1 + (\eta_1^2 - \nu_1^2)rw_1 + 2\eta_1\nu_1 rv_1 = 0; \end{aligned}\right\} \quad 0 < r < r_0$$
$$\left.\begin{aligned} Lv_2 + (\eta_2^2 - \nu_2^2)rv_2 - 2\eta_2\nu_2 rw_2 = 0, \\ Lw_2 + (\eta_2^2 - \nu_2^2)rw_2 + 2\eta_2\nu_2 rv_2 = 0. \end{aligned}\right\} \quad r_0 < r < a \tag{68}$$

From these equations we easily get the relation

$$v_1Lw_1 - w_1Lv_1 + 2\eta_1\nu_1 r(v_1^2 + w_1^2) = 0, \quad 0 < r < r_0$$
$$v_2Lw_2 - w_2Lv_2 + 2\eta_2\nu_2 r(v_2^2 + w_2^2) = 0. \quad r_0 < r < a \tag{69}$$

We first remark that the operator L is singular at $r = 0$. Therefore we consider the improper integral

$$2\eta_1\nu_1 \lim_{\epsilon\to 0} \int_\epsilon^{r_0} (v_1^2 + w_1^2) r \, \mathrm{d}r + 2\eta_2\nu_2 \int_{r_0}^{a} (v_2^2 + w_2^2) r \, \mathrm{d}r$$
$$= \lim_{\epsilon\to 0} \int_\epsilon^{r_0} (w_1Lv_1 - v_1Lw_1) \, \mathrm{d}r + \int_{r_0}^{a} (w_2Lv_2 - v_2Lw_2) \, \mathrm{d}r,$$
$$= \lim_{\epsilon\to 0} [rW[w_1, v_1]]_\epsilon^{r_0} + [rW[w_2, v_2]]_{r_0}^{a}, \tag{70}$$

where $W[w, v] \equiv (wv' - vw')$. First $r \to 0$ as $\epsilon \to 0$; secondly the self-adjoint operator L is *symmetric* and therefore the conjunct of the operator is zero at the boundaries. Hence we get the equation

$$\eta_1\nu_1 \int_0^{r_0} (v_1^2 + w_1^2) r \, \mathrm{d}r + \eta_2\nu_2 \int_{r_0}^{a} (v_2^2 + w_2^2) r \, \mathrm{d}r = 0. \tag{71}$$

Let κ_1 and κ_2 be the wave numbers defined by eqns (9) and (11). Assuming $\xi = \xi_1 + i\xi_2$ in these equations, we get

$$\kappa_1^2 = \omega^2/c_1^2 - \xi^2 = (\omega^2/c_1^2 - \xi_1^2 + \xi_2^2) - 2i\xi_1\xi_2,$$
$$\kappa_2^2 = \omega^2/c_2^2 - \xi^2 = (\omega^2/c_2^2 - \xi_1^2 + \xi_2^2) - 2i\xi_1\xi_2, \tag{72}$$

and therefore comparing with eqn (67) we find that

$$\eta_1\nu_1 = \eta_2\nu_2 = -\xi_1\xi_2. \tag{73}$$

Equation (71) therefore takes the form

$$\xi_1\xi_2 \left\{ \int_0^{r_0} (v_1^2 + w_1^2) r \, \mathrm{d}r + \int_{r_0}^{a} (v_2^2 + w_2^2) r \, \mathrm{d}r \right\} = 0. \tag{74}$$

We first assume that the quantity inside the braces is non-zero. Then eqn (74) will be satisfied if either $\xi_1 = 0$ or $\xi_2 = 0$. This says that real and imaginary wave numbers ξ are admissible. Now let ξ be complex so that ξ_1 and ξ_2 are both non-zero. Then for the equation to hold the quantity within the braces must be zero. Since each of the integrand is positive-definite, it follows that $u_1 = (v_1, w_1) \equiv 0$, and $u_2 = (v_2, w_2) \equiv 0$. This implies that for complex values of ξ, the functions

$u_1(r)$ and $u_2(r)$ are identically zero and therefore complex ξ is inadmissible. We therefore conclude that frequency equation (13) does not admit complex wave number ξ, but can be real or imaginary.

COINCIDENCE OF FREQUENCIES

It is easy to demonstrate that for each frequency $\omega > 0$ there exists only one linearly independent eigenfunction, and therefore all roots of the frequency equation are simple and *coincidence* cannot take place. Suppose that for a given frequency, eqn (66) has two linearly independent eigenfunctions ϕ_1 and ϕ_2. If we compute the conjunct of the operator and use the boundary conditions to evaluate it we find that the conjunct vanishes. However, in this case the conjunct is the Wronskian and vanishing of the Wronskian implies that the two solutions cannot be linearly independent as assumed. Hence all the roots of the frequency equation for $\omega > 0$ are simple and coincidence of frequencies cannot occur, because our problem is of Sturm–Liouville type[13].

On the other hand $\kappa = 0$ when $\omega = \xi = 0$ and the equation degenerates to an Euler's equation. The conjunct of the operator is

$$\underset{\lim \kappa \to 0}{(\mathrm{d}/\mathrm{d}\kappa)}\, W[\phi_1, \phi_2]_{\partial R} = \langle \phi_1, \phi_2 \rangle$$

and hence non-zero. We thus have a problem of co-existence when $\omega = \xi = 0$. In the theory of spectral *representation* this knowledge is of fundamental importance.

VELOCITY OF ENERGY FLOW

In an open connected domain R, the stress equation of motion, correspoding to displacement equation (1), can be written as

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z} - \rho \frac{\partial^2 u}{\partial t^2} = 0, \qquad R\colon\ R_1 \times R_2 \tag{75}$$

where in terms of the physical components

$$\tau_{r\theta} = \mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = 2\mu e_{r\theta}, \qquad \tau_{z\theta} = \mu \frac{\partial u}{\partial z} = 2\mu e_{z\theta}. \tag{76}$$

The total energy is

$$E = \int_R er\,\mathrm{d}r\,\mathrm{d}z, \tag{77}$$

where the scalar energy density is

$$e = (\tau_{r\theta}e_{r\theta} + \tau_{z\theta}e_{z\theta}) + \frac{1}{2}\rho \dot{u}^2, \tag{78}$$

which is the sum of the potential and kinetic energy densities.

The time rate of change of energy density can be easily obtained by differentiating the energy density e with respect to the time parameter t. When the shear stresses satisfy the equation of motion (75), the rate of change of energy density is given by

$$\frac{\partial e}{\partial t} - \frac{1}{r}\left[\frac{\partial}{\partial r}(\dot{u}r\tau_{r\theta}) + \frac{\partial}{\partial z}(\dot{u}r\tau_{z\theta})\right] = 0. \tag{79}$$

Relating the physical components to components of tensor density, we now define

$$re \equiv \bar{e}, \qquad \dot{u}r\tau_{r\theta} \equiv \dot{u}_2\bar{\tau}^{12}, \qquad \dot{u}r\tau_{z\theta} \equiv \dot{u}_2\bar{\tau}^{32}, \tag{80}$$

where u_2 is the covariant component of displacement, $\bar{\tau}^{\kappa 2}$ are the contravariant components of stress tensor density of weight +1 and $\bar{e}$ is the scalar energy density of weight +1. The equation of continuity of energy and its flow now takes a familiar (*covariant*) form

$$\frac{\partial \bar{e}}{\partial t} - \partial_\kappa(\dot{u}_2\bar{\tau}^{\kappa 2}) = 0, \qquad \kappa = 1, 3 \tag{81}$$

where we have used the standard summation convention, identified the coordinates r, θ, z with the indices 1, 2, 3 respectively and ∂_κ represents the partial derivative with respect to the (κ)-th coordinate[14]. On comparing this equation with the flow of matter in fluid, we easily infer that the mean velocity of energy flow is given by

$$C^\kappa_{(e)} = -\frac{\int_0^t \mathrm{d}t \int_R \dot{u}_2\bar{\tau}^{\kappa 2}\,\mathrm{d}\bar{\sigma}}{\int_0^t \mathrm{d}t \int_R \bar{e}\,\mathrm{d}\bar{\sigma}}, \qquad R\colon\ R_1 \times R_2 \tag{82}$$

where $C^\kappa_{(e)}$ is the (κ)-th component of group velocity.

DETAILS OF THE DISPERSION SPECTRUM

As pointed out earlier, the sketching of the spectrum is considerably simplified by the presence of bounding curves corresponding to the value of parameter $Q \to 0^+$ and $Q \uparrow \infty$. The spectral lines must pass through the intersection points corresponding to the family of bounds $Q \to 0^+$, and the family for $Q \uparrow \infty$, separately, because the solution common to the two problems is also a solution to the original problem. In addition, the spectral lines cannot cross these bounds because such solutions do not satisfy eqn (13).

For convenience of drawing the spectrum we introduce non-dimensional frequency Ω and non-dimensional wave number λ, where

$$\Omega \equiv \omega/\bar{\omega}, \qquad \lambda \equiv (a - r_0)\xi/\pi, \tag{83}$$

and $\bar{\omega} \equiv \pi c_2/(a - r_0)$, is the lowest thickness-shear frequency of an infinite plate of thickness $(a - r_0)$, [9]. In terms of non-dimensional variables the bounding curves for $Q \to 0^+$ are

$$\Omega^2 - \lambda^2 = [(1-\eta)/\pi]^2\alpha_q^2, \quad \Omega^2/c_R^2 - \lambda^2 = [(1-\eta)/\pi\eta]^2\gamma_p^2, \quad \left.\begin{matrix} q \\ p \end{matrix}\right\} = 0, 1, 2, \ldots \tag{84}$$

where $c_R^2 \equiv c_1^2/c_2^2 \equiv \sigma/\rho$. The coordinates of the intersection points are

$$\lambda_{pq}^2 = \left(\frac{1-\eta}{\pi\eta}\right)^2 (c_R^2\gamma_p^2 - \eta^2\alpha_q^2)/(1 - c_R^2), \qquad \Omega_{pq}^2 = \left(\frac{1-\eta}{\pi\eta}\right)^2 (\gamma_p^2 - \eta^2\alpha_q^2)c_R^2/(1 - c_R^2). \tag{85}$$

For the second set of bounding curves corresponding to $Q \uparrow \infty$ and the coordinates of the intersection points, we replace γ_p, α_q by γ_m, α_n in the above formulas. The values of γ_p, α_q and γ_m, α_n for $Q \to 0^+$ and $Q \uparrow \infty$, are the simple zeros of eqns (30) and (31), respectively.

In the (Ω, λ)-plane, the straight lines $\Omega = \lambda$ for $\alpha = 0$ and $\Omega = c_R\lambda$ for $\gamma = 0$, are the asymptotes to the bounding curves. These asymptotic lines divide the real plane in three regions, (i) $\Omega \geq \lambda$, (ii) $\lambda < \Omega \leq \lambda c_R$ and (iii) $\lambda c_R < \Omega \leq 0$, where we have assumed that $c_R < 1$. In the first region α and γ are both real; in the second region α is pure imaginary and γ is real; and finally in the third region, α and γ are both pure imaginary. However, as shown earlier, no roots exist in the third region. The asymptotic line $\Omega = \lambda$ intersect the bounds $Q \to 0^+$ and $Q \uparrow \infty$, and their coordinates of intersection are given by eqn (85) if we set $\alpha_0 = 0$ in these equations. The cut-off frequencies $\Omega_{\lambda \to 0}$ are the roots of transcendental equation (13), if in this equation we set $\alpha = \pi\Omega/(1 - \eta)$ and $\gamma = \eta\pi\Omega/[(1 - \eta)c_R]$. In the high frequency region this equation takes the simpler form of eqn (51).

The dispersion spectrum can now be calculated using the above information as guide. For real α and γ, eqn (13) must hold. Choosing appropriate starting values and increments for α, the corresponding γ's may be found by numerical iteration. Each γ can be identified to the mode it represents by means of the bounds on α and γ. Once α and γ are known, the corresponding values of Ω and λ are given by

$$\Omega^2 = \left(\frac{1-\eta}{\pi}\right)^2 [(\gamma_p/\eta)^2 - \alpha_q^2]/(1/c_R^2 - 1), \qquad \lambda^2 = \left(\frac{1-\eta}{\pi}\right)^2 [(c_R\gamma_p/\eta)^2 - \alpha_q^2]/(1/c_R^2 - 1). \quad (86)$$

As soon as the asymptotic line $\Omega = \lambda$ is crossed, α becomes imaginary, γ remains real and now eqn (16) must be satisfied. Once α and γ are determined from this equation, the

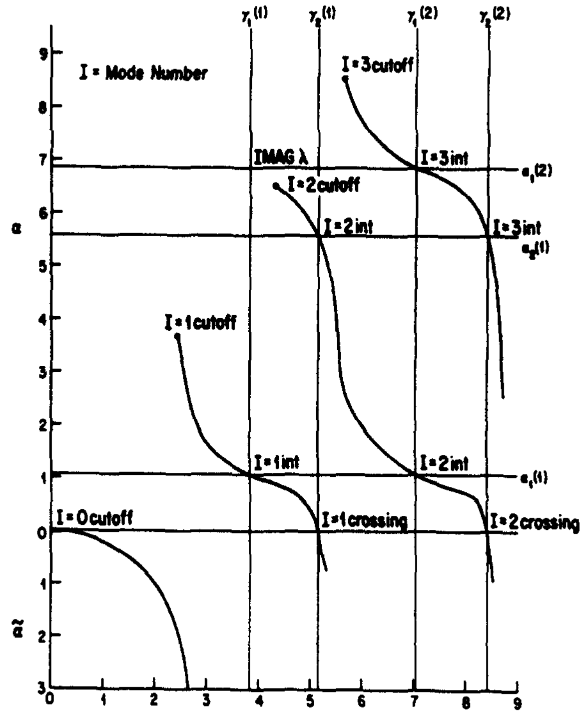

Fig. 1. Dispersion spectrum in (α, γ)-space with bounds and cut-off frequencies.

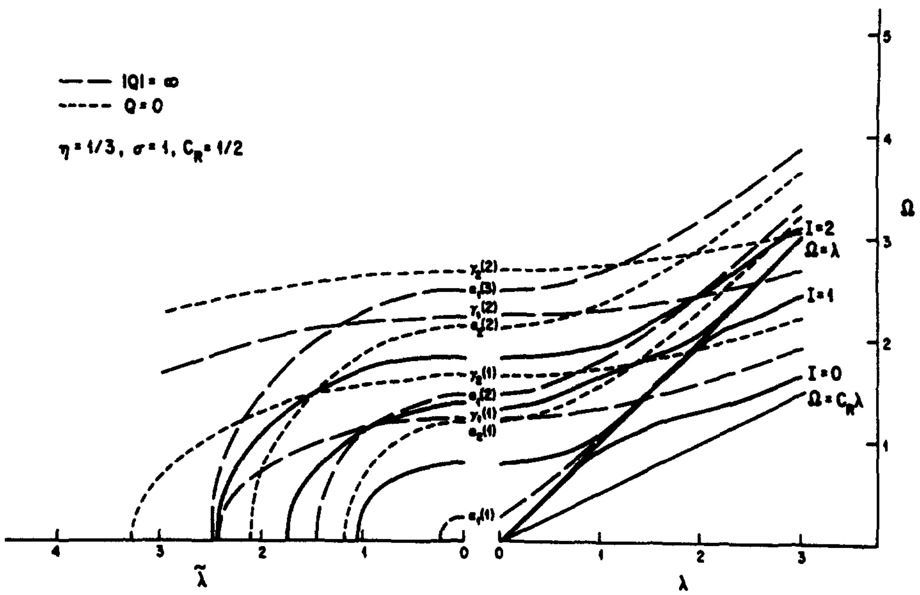

Fig. 2. Dispersion spectrum in (Ω, λ)-space for real and imaginary λ with bounds and asymptotes.

coordinates Ω and λ can be determined from eqn (86), if in this equation we replace α by $i\alpha$. As mentioned earlier, when $c_R < 1$, there are no roots of eqn (18), when both α and γ are imaginary. Hence the spectral lines never intersect the asymptotic line $\Omega = \lambda c_R$, but approach it asymptotically from above. After the solution for real λ has been carried out, the procedure is repeated for imaginary values of λ. The bounding curves are now circles and ellipses, rather than hyperbolas as was the case for real λ. The computational procedure is the same as for real λ, except that we replace λ by $i\lambda$ in all relevant formulas.

Figures 1 and 2 illustrate a typical set of dispersion curves for $\eta = 1/3$, $\sigma = 1$ and $c_R = 1/2$, first in the (α, γ)-space and then in the more familiar (Ω, λ)-space. The subscripts 1, 2 on α and γ indicate the bounding curves corresponding to $Q \uparrow \infty$ and $Q \to 0^+$, respectively. The integer inside the brackets indicates the "mode" or value of p, q, m or n, for example $\alpha_2(3) \equiv \alpha_{q=3}$, $\gamma_1(1) = \gamma_{n=1}$, etc.

REFERENCES

1. W. M. Ewing, W. S. Jardetzky and F. Press, *Elastic Waves in Layered Media*. McGraw-Hill, New York (1957).
2. R. N. Thurston, Elastic waves in rods and clad rods. *J. Acous. Soc. Am.* **64**, 1–37 (1978).
3. J. D. Achenbach, Y. H. Pao and H. F. Tiersten, *Application of Elastic Waves in Electrical Devices*. National Science Foundation, Engineering Mechanical Section (1976).
4. A. E. Armenakas, Torsional waves in composite rods. *J. Acous. Soc. Am.* **38**, 439–446 (1965).
5. R. C. Reuter, Jr., First-branch dispersion of torsional waves in bimaterial rods. *J. Acous. Soc. Am.* **46**, 821–823 (1969).
6. D. W. Haines and P. C. Y. Lee, Axially symmetric torsional waves in circular composite cylinders. *J. Appl. Mech.* **38**, 1042–1044 (1971).
7. H. Kolsky, *Stress Waves in Solids*, pp. 65–68. Oxford University Press, London (1953).
8. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. A.M.S. No. 55. National Bureau of Standards, Washington, D.C. (1964).
9. R. D. Mindlin, *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*. U.S. Army Signal Corps Engineering Laboratories; Fort Monmouth, New Jersey (1955).
10. R. P. Shaw and R. K. Kaul, Interfacial Elastic Parameters in Torsional Vibrations of a Periodic Structured Cylindrical Rod. *Int. J. Solids Structures* **16**, 777–783 (1980).
11. T. J. Delph, G. Herrmann and R. K. Kaul, Coalescence of frequencies and conical points in the dispersion spectra of elastic bodies. *Int. J. Solids Structures* **13**, 423–436 (1977).
12. J. McMahon, On the roots of the Bessel and certain related functions. *Ann. Math.* **9**, 23–30 (1894).
13. G. Birkhoff and G. Rota, *Ordinary Differential Equations*. Ginn, Boston (1962).
14. J. A. Schouten, *Tensor Analysis for Physicists*. Oxford University Press, London (1959).